Calculus Placement Test
Answer Key

I.

1. $f(g(x)) = f(3^x) = (3^x)^2 = 3^{2x}$

   $f(g(2)) = 3^{2(2)} = 3^4 = 81$

2. $f(x) + 2H(x) = x^2 + 2(1 - x^2)$

   $= x^2 + 2 - 2x^2$

   $= 2 - x^2$

3. $\lim_{x \to 0} 3^x = 3^0 = 1$

4. $\lim_{x \to \infty} \frac{H(x)}{f(x)} \lim_{x \to \infty} \frac{1 - x^2}{x^2} = \lim_{x \to \infty} \frac{\frac{1}{x^2} - 1}{1} = -1$

5. $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

   $= \lim_{h \to 0} \frac{2(x+h)^2 - 2x^2}{h}$

   $= \lim_{h \to 0} \frac{(2x^2 + 2xh + h^2) - 2x^2}{h}$

   $= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 2x^2}{h}$

   $= \lim_{h \to 0} \frac{4xh + 2h^2}{h}$

   $= \lim_{h \to 0} (4x + 2h) = 4x$

II.

1. $r = \sqrt{1 - 2\theta}$ $\quad \frac{dr}{d\theta} = \frac{1}{2}(1 - 2\theta)^{-\frac{1}{2}}(-2)$

   $\frac{dr}{d\theta} = \frac{-1}{\sqrt{1 - 2\theta}}$

2. $w = xe^{2x}$ $\quad \frac{dw}{dx} = x(2e^{2x}) + e^{2x}(1)$

   $= e^{2x}(2x + 1)$

3. $y = \ln(\sin(\theta))$ $\quad \frac{dy}{d\theta} = \frac{1}{\sin(\theta)} \cdot \cos(\theta)$

   $= \cot(\theta)$

III.

1. Let $u = \ln(x)$; $du = \frac{1}{x} dx$

   $\int_1^{e^2} \frac{\sqrt{\ln(x)}}{x} dx = \int_1^{e^2} u^{\frac{1}{2}} du$

   $= \left. \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right|_1^{e^2}$

   $= \left. \frac{2}{3}(\ln(x))^{\frac{3}{2}} \right|_1^{e^2}$

   $= \frac{2}{3}\left[(\ln(e^2))^{\frac{3}{2}} - (\ln(1))^{\frac{3}{2}}\right]$

   $= \frac{2}{3}(\sqrt{8})$

   $= \frac{2}{3}(2\sqrt{2}) = \frac{4\sqrt{2}}{3}$

2. Let $u = \sin(2x)$; $du = \cos(2x) \cdot dx \cdot 2$

   $\frac{1}{2} \int \sin(2x)\cos(2x) dx \cdot 2$

   $= \frac{1}{2} \int u du$

   $= \frac{1}{2} \frac{u^2}{2} + C$

   $= \frac{1}{4}(\sin^2(2x)) + C$

IV.

$f(x) = -2x^3 + 6x^2 - 3$

$f'(x) = -6x^2 + 12x$

$-6x(x - 2) = 0$

$x = 0$, $x = 2$ are critical points.

$f''(x) = -12x + 12$

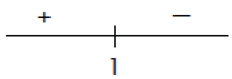
$-12(x - 1) = 0$

$x = 1$ possible inflection point

$f''(0) = 12$ so $x = 0$ is a min.

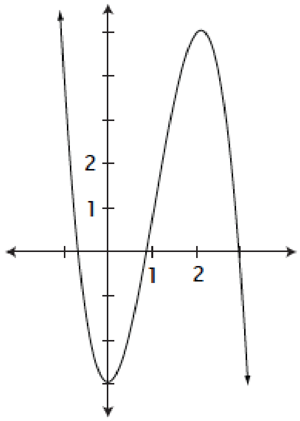
$f''(2) = -12$ so $x = 2$ is a max.

f''



$x = 1$ is an inflection point

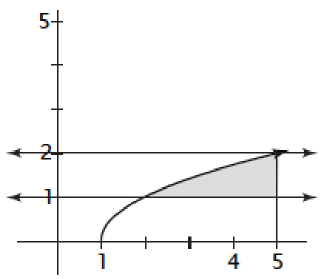
x	y
0	−3
1	1
2	5
3	−3
−1	5



f is concave up from $(-\infty, 1)$
and concave down from $(1, \infty)$.

V. $y = \sqrt{x-1}$

$y^2 = x - 1;\ x = y^2 + 1$



$$\int_1^2 (y^2+1)\,dy = \left(\frac{y^3}{3} + y\right)\Bigg|_1^2$$

$$= \left(\frac{8}{3} + 2\right) - \left(\frac{1}{3} + 1\right) = \frac{10}{3}$$

VI.

1. $\lim_{x\to 3} \frac{x^2-9}{x^2+x} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x(x+1)} = 0$

2. $\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}} = 6$

3. $\lim_{x\to 0} \frac{\cos(x)-1}{\sin(x)} = \lim_{x\to 0} \frac{-\sin(x)}{\cos(x)} = 0$ using LR

4. $\lim_{x\to\infty} \left(1 + \frac{5}{x}\right)^{2x}$

   $\frac{5}{x} \to 0$ so $\lim_{x\to\infty} 1^{2x} = 1$

5. $\lim_{x\to\infty} \frac{3x^2-4x+7}{x^2+3} = \lim_{x\to\infty} \frac{3 - \frac{4}{x} + \frac{7}{x^2}}{1 + \frac{3}{x^2}} = 3$

VII. $f(x) = x^2 - \ln(x)$

$f'(x) = 2x - \frac{1}{x}$

$f'(1) = 1$

$y = mx + b$

$1 = 1(1) + b$

$b = 0$

So the equation of the tangent line is $y = x$.

VIII. $y = x^2 + 1;\ y = x + 3$

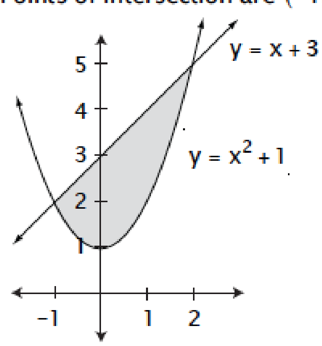
$x^2 + 1 = x + 3$

$x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

$x = -1,\ 2$

Points of intersection are $(-1, 2)$ and $(2, 5)$.



$$\int_{-1}^{2}\left[(x+3)-(x+1)\right]dx$$

$$= \int_{-1}^{2}\left(-x^2+x+2\right)dx$$

$$= \left(-\frac{x^3}{3}+\frac{x^2}{2}+2x\right)\Bigg|_{-1}^{2}$$

$$= \left(-\frac{8}{3}+2+4\right)-\left(\frac{1}{3}+\frac{1}{2}-2\right)$$

$$= -\frac{8}{3}-\frac{5}{6}+8 = 4\frac{1}{2}$$

IX. $2x+4y=16$

$$y = 4-\frac{1}{2}x$$

$$A(x) = xy$$

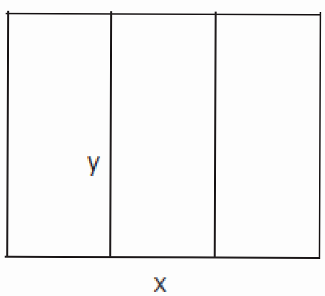
$$= x\left(4-\frac{1}{2}x\right)$$

$$= 4x-\frac{1}{2}x^2$$

$A'(x) = 4-x = 0$ when $x = 4$

$A''(x) = -1$

so $A''(4) = -1$ which yields a max.



If $x = 4$, when $y = 4-\frac{1}{2}(4) = 2$.

The dimensions of the largest possible area would be 4 ft by 2 ft.